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BLASIUS SERIES FOR HEAT AND MASS TRANSFER

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NOMENCLATURE

c_i,	concentration of diffusing species;
c_0,	concentration at solid surface;
c_∞,	concentration far from the surface;
D,	diffusion coefficient;
$f_1, f_3 \ldots$,	coefficients in Blasius series for stream function;
N_y,	y-component of the flux of the diffusing species;
Sc,	$= \nu/D$, the Schmidt number;
$u_1, u_3 \ldots$,	coefficients in power-series expansion of U;
U,	velocity at outer edge of boundary layer;
v_x, v_y,	components of the velocity;
x,	distance from the stagnation point along the surface;
y,	perpendicular distance from the surface;
β,	velocity derivative at the surface;
$\Gamma(\frac{4}{3})$,	$= 0.89298$;
η,	dimensionless distance from the surface;
Θ,	dimensionless concentration;
$\Theta_0, \Theta_2 \ldots$,	coefficients in Blasius series for mass transfer;
ν,	kinematic viscosity;
ψ,	stream function.

INTRODUCTION

DIFFERENTIAL equations describing fluid flow and mass transfer in two-dimensional boundary layers are (see, for example, Schlichting [1], pp. 110–112)

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{\mathrm{d}U}{\mathrm{d}x} + \nu \frac{\partial^2 v_x}{\partial y^2}. \quad (1)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \quad (2)$$

$$v_x \frac{\partial c_i}{\partial x} + v_y \frac{\partial c_i}{\partial y} = D \frac{\partial^2 c_i}{\partial y^2}. \quad (3)$$

The boundary conditions for a constant concentration on the solid surface are

$$\left.\begin{aligned} c_i &= c_0, \quad v_x = v_y = 0 \quad \text{at } y = 0. \\ c_i &= c_\infty, \quad v_x = U(x) \quad \text{at } y = \infty. \end{aligned}\right\} \quad (4)$$

The velocity $U(x)$ is the inner limit of the appropriate outer (Euler) solution which describes the fluid motion outside the boundary layer.

If viscous dissipation is ignored, the same equations would apply to heat transfer in boundary layers, in which case c_i should be replaced by the temperature T and D replaced by the thermal diffusivity $\alpha = k/\rho\hat{C}_p$. In either case the pertinent physical properties are taken to be constant.

The specified boundary conditions on the solid surface are not completely general. Nevertheless, a constant concentration on the surface is commonly encountered in mass-transfer problems as well as in the analogous heat-transfer problems and provides a useful starting point. The consideration of an arbitrary variation of concentration with position on the surface involves a higher order of complexity.

The normal component v_y of the velocity is taken to be zero even though a non-zero mass-transfer rate implies a non-zero interfacial velocity. The consequences of such an interfacial velocity have been treated by Stewart [2] for flow past a flat plate at zero incidence, by Olander [3] for a rotating disk, and by Acrivos [4] for boundary-layer mass transfer at infinite Schmidt number. In the absence of calculated results of the effect of an interfacial velocity applicable to a specific situation, the mass-transfer rate calculated for $v_y = 0$ can be corrected by a factor based on the papers cited.

Solutions for the boundary-layer problem stated in equations (1)–(4) can be obtained by a number of methods (see Schlichting [1]). Exact solutions have been obtained for certain restricted geometries. Approximate methods have also been developed which are less restricted but also less reliable. In many cases the validity of approximate methods is judged by a comparison with the exact solutions.

One class of exact solutions involves series expansions in terms of "universal" functions which can be tabulated. These functions are universal in the sense that they are defined so as to be independent of the specific form of the function $U(x)$. How this is done can be seen by referring ahead to equations (6), (9), (10) and (11).

The Blasius series is perhaps the simplest form of a series solution and involves Taylor series expansions in x about the stagnation point or the leading edge of the body. This procedure was suggested by Blasius in 1908. Other series, such as that of Görtler [5], involve expansions in more complicated functions of x.

VELOCITY PROFILES

For symmetric, two-dimensional flow past a cylindrical surface with a rounded nose, the external velocity $U(x)$ can be expressed as a power series:

$$U(x) = u_1x + u_3x^3 + u_5x^5 + u_7x^7 + \ldots \tag{5}$$

Because of the assumed symmetry about a plane parallel to the undisturbed flow, only the odd powers of x are present.

The Blasius series then expresses the stream function $\psi(x, y)$ also as a power series in x:

$$\psi = \sqrt{[\nu/u_1]}\,\{u_1xf_1(\eta) + 4u_3x^3f_3(\eta) + 6u_5x^5f_5(\eta) + 8u_7x^7f_7(\eta) + 10u_9x^9f_9(\eta) + 12u_{11}x^{11}f_{11}(\eta) + \ldots\}, \tag{6}$$

where

$$\eta = y\sqrt{(u_1/\nu)}, \tag{7}$$

and where ψ is related to the velocity components by the equations

$$v_x = \frac{\partial\psi}{\partial y}, \quad v_y = -\frac{\partial\psi}{\partial x}. \tag{8}$$

One wants to tabulate functions independent of u_1, u_3, etc. The first two functions, f_1 and f_3, satisfy this criterion, but f_5, f_7, etc., do not. Hence, it is necessary to split up the higher order terms. For example,

$$\left.\begin{aligned} f_5 &= g_5 + \frac{u_3^2}{u_1u_5}h_5,\\ f_7 &= g_7 + \frac{u_3u_5}{u_1u_7}h_7 + \frac{u_3^3}{u_1^2u_7}k_7. \end{aligned}\right\} \tag{9}$$

The universal functions thus are f_1, f_3, g_5, h_5, etc., and have been calculated notably by Tifford [6]. Such tables are reproduced by several authors (Schlichting [1], pp. 150–151, Curle [7], p. 26). Because it is a power series in x, the Blasius series is most useful in the region near the stagnation point.

CONCENTRATION PROFILES

In a similar manner the concentration (or the temperature) can be expanded in a power series (see, for example, Schlichting [1], p. 319):

$$\Theta(x, y) = \frac{c_i - c_0}{c_\infty - c_0} = \Theta_0(\eta) + \frac{u_3}{u_1}x^2\Theta_2(\eta) + \frac{u_5}{u_1}x^4\Theta_4(\eta) + \ldots \tag{10}$$

The functions Θ_0 and Θ_2 are "universal" functions independent of u_1, u_3, etc., although they do depend on the Schmidt number. In order to form universal functions, the higher order coefficients must be broken up in a manner similar to the hydrodynamic functions [compare equations (9)]:

$$\left.\begin{aligned} \Theta_4 &= a_4 + \frac{u_3^2}{u_1u_5}b_4,\\ \Theta_6 &= a_6 + \frac{u_3u_5}{u_1u_7}b_6 + \frac{u_3^3}{u_1^2u_7}d_6,\\ \Theta_8 &= a_8 + \frac{u_3u_7}{u_1u_9}b_8 + \frac{u_5^2}{u_1u_9}d_8 + \frac{u_3^2u_5}{u_1^2u_9}e_8 + \frac{u_3^4}{u_1^3u_9}p_8,\\ \Theta_{10} &= a_{10} + \frac{u_3u_9}{u_1u_{11}}b_{10} + \frac{u_5u_7}{u_1u_{11}}d_{10} + \frac{u_3^2u_7}{u_1^2u_{11}}e_{10} + \frac{u_3u_5^2}{u_1^2u_{11}}p_{10} + \frac{u_3^3u_5}{u_1^3u_{11}}r_{10} + \frac{u_3^5}{u_1^4u_{11}}s_{10}. \end{aligned}\right\} \tag{11}$$

The differential equations for some of the universal mass-transfer, boundary-layer functions are

$$\left.\begin{aligned} (1/Sc)\,\Theta_0'' + f_1\Theta_0' &= 0,\\ (1/Sc)\,\Theta_2'' + f_1\Theta_2' - 2f_1'\Theta_2 &= -12f_3\Theta_0',\\ (1/Sc)\,a_4'' + f_1a_4' - 4f_1'a_4 &= -30g_5\Theta_0',\\ (1/Sc)\,b_4'' + f_1b_4' - 4f_1'b_4 &= -12f_3\Theta_2' + 8f_3'\Theta_2 - 30h_5\Theta_0', \end{aligned}\right\} \tag{12}$$

and the corresponding boundary conditions are

$$\left.\begin{aligned} &\Theta_0 = \Theta_2 = a_4 = b_4 = 0 &&\text{at } \eta = 0,\\ &\Theta_0 = 1,\quad \Theta_2 = a_4 = b_4 = 0 &&\text{at } \eta = \infty. \end{aligned}\right\} \tag{13}$$

The functions f_1, f_3, g_5 and h_5 appearing in equations (12) are the same universal functions defined in equations (6) and (9).

MASS-TRANSFER COEFFICIENTS

The differential equations for the nineteen universal functions were solved numerically. Since the universal mass-transfer, boundary-layer functions depend upon Sc as well as on η, their tabulation could become unwieldy. The local rate of mass transfer is of interest and is given by

$$N_y(x) = -D\frac{\partial c_i}{\partial y}\bigg|_{y=0} = -D(c_\infty - c_0)\sqrt{\left[\frac{u_1}{\nu}\right]}\Big\{\Theta_0'(0) + \frac{u_3}{u_1}x^2\Theta_2'(0) + \frac{u_5}{u_1}x^4\Theta_4'(0) + \frac{u_7}{u_1}x^6\Theta_6'(0) + \frac{u_9}{u_1}x^8\Theta_8'(0) + \ldots\Big\} \tag{14}$$

The coefficients necessary for the calculation of the rate of mass transfer are given in Table 1. For heat transfer, the Prandtl number replaces the Schmidt number. The range of Schmidt numbers in Table 1 was selected to cover heat- and mass-transfer problems of practical interest. For heat transfer in molten metals the Prandtl number is near 0·01, for gases it is near 0·7, and for other liquids it ranges from 5 to several hundred. For mass transfer in gases the Schmidt number is near 1, and in liquids it ranges from several hundred to several thousand. Some idea of the accuracy of the numbers in Table 1 can be gained from a comparison of the results with a basic mesh size of 0·02 with those for a basic mesh size of 0·0108. These suggest that the errors of the numbers in Table 1 are less than 0·01 per cent for small Schmidt numbers (through $Sc = 1$) but increase to about 0·1 or 0·2 per cent for $Sc = 1000$.

Asymptotic forms of the coefficients for large and for small

Table 1. Dimensionless mass-transfer coefficients from Blasius series

Sc	$\Theta_0'(0)$	$\Theta_2'(0)$	$a_4'(0)$	$b_4'(0)$	$a_6'(0)$	$b_6'(0)$	$d_6'(0)$
0·005	0·0545	0·0424	0·0492	−0·0114	0·0539	−0·0264	0·0083
0·01	0·0760	0·0598	0·0704	−0·0171	0·0781	−0·0414	0·0138
0·02	0·1054	0·0844	0·1009	−0·0260	0·1134	−0·0652	0·0228
0·05	0·1610	0·1323	0·1619	−0·0450	0·1856	−0·1177	0·0433
0·10	0·2195	0·1847	0·2304	−0·0675	0·2678	−0·1808	0·0682
0·20	0·2964	0·2557	0·3250	−0·0996	0·3828	−0·2718	0·1042
0·50	0·4334	0·3866	0·5025	−0·1613	0·6006	−0·4479	0·1741
0·70	0·4959	0·4476	0·5859	−0·1906	0·7036	−0·5319	0·2075
1·00	0·5705	0·5210	0·6868	−0·2262	0·8284	−0·6341	0·2482
2·00	0·7437	0·6931	0·9246	−0·3106	1·1235	−0·8772	0·3450
5·00	1·0435	0·9937	1·3418	−0·4595	1·6427	−1·3075	0·5169
10·00	1.3389	1·2911	1·7556	−0·6077	2·1587	−1·7372	0·6890
20·00	1·7104	1·6656	2·2772	−0·7950	2·8098	−2·2809	0·9071
50·00	2·3529	2·3132	3·1797	−1·1196	3·9367	−3·2245	1·2863
100·00	2·9869	2·9518	4·0698	−1·4402	5·0484	−4·1573	1·6613
200·00	3·7855	3·7556	5·1899	−1·8441	6·4474	−5·3330	2·1340
500·00	5·1685	5·1454	7·1265	−2·5436	8·8660	−7·3708	2·9530
1000·00	6·5353	6·5160	9·0355	−3·2348	11·2500	−9·3864	3·7622

Sc	$a_8'(0)$	$b_8'(0)$	$d_8'(0)$	$e_8'(0)$	$p_8'(0)$
0·005	0·0577	−0·0304	−0·0160	0·0339	−0·0110
0·01	0·0846	−0·0488	−0·0263	0·0582	−0·0193
0·02	0·1242	−0·0784	−0·0429	0·0981	−0·0329
0·05	0·2064	−0·1443	−0·0804	0·1888	−0·0640
0·10	0·3011	−0·2241	−0·1259	0·2993	−0·1016
0·20	0·4345	−0·3393	−0·1919	0·4591	−0·1559
0·50	0·6889	−0·5628	−0·3201	0·7691	−0·2610
0·70	0·8095	−0·6696	−0·3814	0·9173	−0·3113
1·00	0·9560	−0·7998	−0·4562	1·0981	−0·3726
2·00	1·3032	−1·1096	−0·6342	1·5291	−0·5190
5·00	1·9156	−1·6595	−0·9505	2·2966	−0·7803
10·00	2·5249	−2·2094	−1·2670	3·0665	−1·0429
20·00	3·2943	−2·9061	−1·6681	4·0438	−1·3768
50·00	4·6266	−4·1162	−2·3652	5·7439	−1.9581
100·00	5·9412	−5·3130	−3·0547	7·4269	−2·5340
200·00	7·5956	−6·8223	−3·9244	9·5495	−3·2605
500·00	10·4559	−9·4393	−5·4326	13·2286	−4·5199
1000·00	13·2752	−12·0294	−6·9254	16·8658	−5·7647

Table 1 continued

Sc	$a'_{10}(0)$	$b'_{10}(0)$	$d'_{10}(0)$	$e'_{10}(0)$	$p'_{10}(0)$	$r'_{10}(0)$	$s'_{10}(0)$
0·005	0·0527	−0·0344	−0·0383	0·0429	0·0487	−0·0657	0·0189
0·01	0·0849	−0·0561	−0·0642	0·0742	0·0849	−0·1157	0·0334
0·02	0·1315	−0·0911	−0·1065	0·1259	0·1448	−0·1981	0·0571
0·05	0·2251	−0·1696	−0·2024	0·2431	0·2806	−0·3840	0·1103
0·10	0·3317	−0·2647	−0·3191	0·3855	0·4455	−0·6087	0·1744
0·20	0·4822	−0·4022	−0·4883	0·5912	0·6838	−0·9322	0·2663
0·50	0·7704	−0·6691	−0·8173	0·9902	1·1460	−1·5583	0·4441
0·70	0·9076	−0·7969	−0·9749	1·1812	1·3673	−1·8581	0·5292
1·00	1·0742	−0·9526	−1·1672	1·4142	1·6373	−2·2241	0·6332
2·00	1·4699	−1·3238	−1·6257	1·9705	2·2824	3·0993	0·8822
5·00	2·1688	−1·9836	−2·4416	2·9623	3·4333	−4·6637	1·3280
10·00	2·8652	−2·6442	−3·2594	3·9585	4·5899	−6·2384	1·7774
20·00	3·7449	−3·4818	−4·2969	5·2241	6·0597	−8·2420	2·3498
50·00	5·2690	−4·9374	−6·1009	7·4271	8·6190	−11·7341	3·3483
100·00	6·7730	−6·3776	−7·8865	9·6088	11·1540	−15·1951	4·3385
200·00	8·6661	−8·1944	−10·1394	12·3614	14·3528	−19·5635	5·5887
500·00	11·9391	−11·3457	−14·0475	17·1337	19·8993	−27·1387	7·7569
1000·00	15·1652	−14·4658	−17·9175	21·8536	25·3852	−34·6297	9·9010

Table 2. Asymptotes for large Schmidt numbers

$\Theta'_0(0) =$	$0{\cdot}6608\ Sc^{1/3}$
$\Theta'_2(0) =$	$0{\cdot}6658\ Sc^{1/3}$
$a'_4(0) =$	$0{\cdot}9280\ Sc^{1/3}$
$b'_4(0) =$	$-0{\cdot}3339\ Sc^{1/3}$
$a'_6(0) =$	$1{\cdot}1592\ Sc^{1/3}$
$b'_6(0) =$	$-0{\cdot}9719\ Sc^{1/3}$
$d'_6(0) =$	$0{\cdot}3917\ Sc^{1/3}$
$a'_8(0) =$	$1{\cdot}3711\ Sc^{1/3}$
$b'_8(0) =$	$-1{\cdot}2475\ Sc^{1/3}$
$d'_8(0) =$	$-0{\cdot}7190\ Sc^{1/3}$
$e'_8(0) =$	$1{\cdot}7590\ Sc^{1/3}$
$p'_8(0) =$	$-0{\cdot}6028\ Sc^{1/3}$
$a'_{10}(0) =$	$1{\cdot}5690\ Sc^{1/3}$
$b'_{10}(0) =$	$-1{\cdot}5016\ Sc^{1/3}$
$d'_{10}(0) =$	$-1{\cdot}8628\ Sc^{1/3}$
$e'_{10}(0) =$	$2{\cdot}2556\ Sc^{1/3}$
$p'_{10}(0) =$	$2{\cdot}6516\ Sc^{1/3}$
$r'_{10}(0) =$	$-3{\cdot}6335\ Sc^{1/3}$
$s'_{10}(0) =$	$1{\cdot}0506\ Sc^{1/3}$

Table 3. Asymptotes for small Schmidt numbers

$\Theta'_0(0) =$	$0{\cdot}7979\ Sc^{1/2}$
$\Theta'_2(0) =$	$0{\cdot}5984\ Sc^{1/2}$
$a'_4(0) =$	$0{\cdot}6649\ Sc^{1/2}$
$b'_4(0) =$	$-0{\cdot}1247\ Sc^{1/2}$
$a'_6(0) =$	$0{\cdot}6981\ Sc^{1/2}$
$b'_6(0) =$	$-0{\cdot}2327\ Sc^{1/2}$
$d'_6(0) =$	$0{\cdot}0436\ Sc^{1/2}$
$a'_8(0) =$	$0{\cdot}7181\ Sc^{1/2}$
$b'_8(0) =$	$-0{\cdot}2244\ Sc^{1/2}$
$d'_8(0) =$	$-0{\cdot}0997\ Sc^{1/2}$
$e'_8(0) =$	$0{\cdot}1122\ Sc^{1/2}$
$p'_8(0) =$	$-0{\cdot}0175\ Sc^{1/2}$
$a'_{10}(0) =$	$0{\cdot}7314\ Sc^{1/2}$
$b'_{10}(0) =$	$-0{\cdot}2194\ Sc^{1/2}$
$d'_{10}(0) =$	$-0{\cdot}1828\ Sc^{1/2}$
$e'_{10}(0) =$	$0{\cdot}1029\ Sc^{1/2}$
$p'_{10}(0) =$	$0{\cdot}0914\ Sc^{1/2}$
$r'_{10}(0) =$	$-0{\cdot}0571\ Sc^{1/2}$
$s'_{10}(0) =$	$0{\cdot}0075\ Sc^{1/2}$

Schmidt numbers are given in Tables 2 and 3, respectively. These are calculated from equations (15) and (17) (see, for example, Acrivos [8]):

For $Sc \to \infty$,

$$\left.\frac{\partial c_i}{\partial y}\right|_{y=0} = \frac{c_\infty - c_0}{\Gamma(\frac{4}{3})(9D)^{1/3}} \frac{\sqrt{[\beta(x)]}}{[\int_0^x (\sqrt{\beta})\,\mathrm{d}x]^{1/3}} \tag{15}$$

where

$$\beta = \left.\frac{\partial v_x}{\partial y}\right|_{y=0} \tag{16}$$

For $Sc \to 0$,

$$\left.\frac{\partial c_i}{\partial y}\right|_{y=0} = \frac{c_\infty - c_0}{\sqrt{(\pi D)}} \frac{U(x)}{[\int_0^x U\,\mathrm{d}x]^{1/2}} \tag{17}$$

Note that when the Schmidt number is small, it is still necessary for the product of the Reynolds number and the Schmidt number to be large in order for the boundary-layer form of the equation of convective diffusion [equation (3)] to be applicable. Otherwise the diffusion layer is too thick, and more of the outer Euler solution is needed than that represented by $U(x)$.

The only peculiar feature of the results is that the mass-transfer coefficients, except $\Theta'_0(0)$, approach the low Sc asymptote from above rather than below. For many of the coefficients the low Schmidt number asymptote is hardly applicable even at $Sc = 0{\cdot}01$, and it was necessary to extend the numerical calculations to $Sc = 10^{-6}$ in order to verify these asymptotes. Therefore a simple interpolation between the low and high Sc asymptotes such as that suggested by Acrivos [8] appears to become more inaccurate for the higher coefficients.

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